Physics 798C Spring24 Superconductivity
Homework 6
Due March 28, 2024

1. Recall that we utilized the Bogoliubov-Valatin transformation to diagonalize the BCS model Hamiltonian:

$$\gamma_{k0}^{+} = u_k^* \, c_{k\uparrow}^{+} - v_k^* \, c_{-k\downarrow} \text{ and } \gamma_{k1}^{+} = u_k^* \, c_{-k\downarrow}^{+} + v_k^* \, c_{k\uparrow},$$

thus defining operators that create Bogoliubons (quasiparticles). Show the following two results through explicit calculations using the BCS ground state wavefunction in the form $|\Psi_{BCS}\rangle = \prod_{l=l_1}^{l_M} (u_l + v_l c_{l\uparrow}^{+} c_{-l\downarrow}^{+}) \, |0\rangle$:

a) $\gamma_{k0}|\Psi_{BCS}\rangle = 0$, i.e. the BCS ground state is the vacuum state for quasiparticles,

b) $\gamma_{k0}^{+}|\Psi_{BCS}\rangle = c_{k\uparrow}^{+} \prod_{l\neq k}^{l_M} (u_l + v_l c_{l\uparrow}^{+} c_{-l\downarrow}^{+}) \, |0\rangle$, i.e. the γ_{k0}^{+} operator creates a single $(k,\uparrow)$ electron with probability 1, and leaves behind a handicapped BCS ground state for the remaining electrons.

2. The Ginzburg-Landau differential equations can be used *above* T_c by having $\alpha(T)$ positive (it is negative below T_c) and, since Ψ will be small above T_c, dropping the cubic term. Suppose that $\Psi = \Psi_o$ at $x = 0$, and the material fills the region $x > 0$. Let $\mathbf{A} = 0$. Show that Ψ decays exponentially away from the boundary, with a characteristic length

$$\xi = [\hbar^2 / 2m^*\alpha(T)]^{1/2}$$

This is in contrast to the behavior below T_c, where spatially-extended order parameters are possible.

This situation occurs even when the "superconductor" has $T_c = 0$, in which case the coherence length is called ξ_N, the normal-metal coherence length. The boundary condition $\Psi = \Psi_o$ at $x = 0$ can be created by having the normal metal be in electrical contact with a superconductor. This is the origin of the *proximity effect*, where superconductivity is induced in a normal metal by proximity to a superconductor. The superconducting electrons (or really a "pairing potential") have "leaked" from the superconductor into the normal metal.

3. A superconducting film is infinite in the *y* and *z* directions, and has thickness *d* in the *x* direction. It obeys the linear GL equation,

$$\frac{\partial^2 \psi}{\partial x^2} = -\frac{\psi}{\xi_{GL}^2(T)}$$

The left side of the film faces vacuum, and the right side is in contact with an infinite normal metal, so the boundary conditions are:

$$\left.\frac{\partial \psi}{\partial x}\right|_{x=0} = 0 \qquad \left.\frac{\partial \psi}{\partial x}\right|_{x=d} = -\frac{\psi}{b},$$

where $b > 0$ is the extrapolation length. Find $T_c^{bilayer}$ of the bilayer as a function of *d*. (You will have to solve a transcendental equation by graphical or other means. Choose the solution for $\xi = \xi_0/[1 - T_c^{bilayer}/T_c^{Bulk}]^{1/2}$ that gives the highest $T_c^{bilayer}$.) Give explicit equations for $d >> b$ and $d << b$. In the latter case, $T_c^{bilayer} = 0$ is not acceptable, the answer must include the next order term.

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4. Consider a <u>normal metal</u> extending from $-d/2$ to $+d/2$ which is described by the linearized GL equation with $\boldsymbol{A} = 0$,

$$\frac{\partial^2 \psi}{\partial x^2} = +\frac{\psi}{\xi_{GL}^2(T)}.$$

Why is there a "+" sign in this equation compared to that in problem 2, above? The normal metal is sandwiched between two superconductors, which impose the boundary conditions:

$$\psi = \psi_L \quad \text{at } x = -d/2$$
$$\psi = \psi_R e^{i\gamma} \quad \text{at } x = +d/2,$$

where ψ_L and ψ_R are real. Show that this leads to a solution in the normal metal which carries a supercurrent $J_s = \frac{q^* \hbar}{m^*} Re[\frac{1}{i}\psi^* \nabla \psi]$, with $J_s = J_c \sin\gamma$, and find an explicit expression for J_c. How does J_c depend on d/ξ_N? The equation for J_s was first derived by Josephson for tunnel junctions using microscopic theory, but applies to superconducting – normal – superconducting (SNS) junctions as well.